An example of steady laminar flow at large Reynolds number

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(Received 22 July 1960)

The purpose of this note is to describe a particular class of steady fluid flows, for which the techniques of classical hydrodynamics and boundary-layer theory determine uniquely the asymptotic flow for large Reynolds number for each of a continuously varied set of boundary conditions. The flows involve viscous layers in the interior of the flow domain, as well as boundary layers, and the investigation is unusual in that the position and structure of all the viscous layers are determined uniquely. The note is intended to be an illustration of the principles that lead to this determination, not a source of information of practical value.

The flows take place in a two-dimensional channel with porous walls through which fluid is uniformly injected or extracted. When fluid is extracted through both walls there are boundary layers on both walls and the flow outside these layers is irrotational. When fluid is extracted through one wall and injected through the other, there is a boundary layer only on the former wall and the inviscid rotational flow outside this layer satisfies the no-slip condition on the other wall. When fluid is injected through both walls there are no boundary layers, but there is a viscous layer in the interior of the channel, across which the second derivative of the tangential velocity is discontinuous, and the position of this layer is determined by the requirement that the inviscid rotational flows on either side of it must satisfy the no-slip conditions on the walls.

1. Introduction

One of the central problems of hydrodynamics which remains substantially unsolved is how to determine the steady flow of a viscous incompressible fluid when the Reynolds number is large. The difficulty cannot fairly be said to be due to the largeness of the Reynolds number. Into the problem as a whole, this largeness introduces a substantial simplification, without which the hopelessly intractable equations of general viscous flow would govern the motion. The point is more that the largeness of the Reynolds number also introduces, as a by-product, a peculiar difficulty which has not been resolved.

The simplification is that the asymptotic flow pattern for very large Reynolds numbers is divided into a number of regions, in the interior of which the relatively simple laws of inviscid dynamics are valid. Moreover, near the bounding surfaces of these regions, where the space gradients of the flow variables are large and viscosity is important, the dynamics again takes a relatively simple form: that involved in boundary-layer theory. These is no proof of these results; but they are, with good reason, commonly accepted and are here taken as the starting-point.

The peculiar difficulty of determining such a flow is that the position of the viscous layers in the flow field is *a priori* unknown. If all the layers were attached to the rigid boundaries of the flow, forming conventional boundary layers, the problem would be relatively easy. This is rarely the case, however; there are usually internal viscous layers, and owing to the non-linearity of the governing equations the position of these layers depends upon the details of the particular solution that is relevant to the problem in hand. An important example of such layers is provided by the phenomenon of flow separation (boundary-layer separation), and it is a sobering thought that, despite the fifty or so years that the techniques of boundary-layer theory have been at our disposal, so little progress has been made with the problem that even such a bulk characteristic as the dependence of drag coefficient on (large) Reynolds number is still a matter of pure conjecture.

Unfortunately, the work described in this note is far too special to be of much, if any, value as a contribution to the general problem described above. But it is an example, I think the first, of a non-linear boundary-value problem of viscous flow theory in which the positions of all the viscous layers, both boundary layers and internal layers, can be deduced unambiguously by the established techniques of boundary-layer theory. As such, the reader may find it interesting and perhaps useful for didactic purposes.

2. Description of the flow

The flow to be studied takes place in a two-dimensional channel bounded by the plane porous walls $y = \pm h$. Through these walls fluid is injected or extracted with constant and uniform velocity V_1 at $y = -h$ and V_2 at $y = +h$. Both of these velocities may have either sign (counted positive in the positive y-direction), and the object is to find the steady laminar flow induced by all possible combinations of V_1 and V_2 consistent with the Reynolds number being large. In this context the Reynolds number is that formed from h, the kinematic viscosity ν, and the greater of the absolute magnitudes of V_1 and V_2.*

Flows of this type seem first to have been studied by Berman (1953) who noticed that the boundary conditions and equations of motion may all be satisfied by assuming that the component of velocity v normal to the walls is independent of x, the co-ordinate measured along the length of the channel. By continuity, the component of velocity u parallel to the walls is then given by

$$u(x, \eta) = -\frac{x}{h} v'(\eta) + u_0(\eta),$$

where the prime denotes differentiation with respect to the non-dimensional variable $\eta = y/h$, and u_0 is an arbitrary function. For such a flow, the equation

* In all viscous flows in long and narrow regions it is the Reynolds number based on the cross flow that is the dynamically significant parameter, because this determines the departure from unidirectional flow in which the acceleration is zero.

of motion normal to the walls may be integrated immediately to give the kinematic pressure p:

$$p = \frac{\nu}{h}v' - \tfrac{1}{2}v^2 + P(x),$$

where $P(x)$ is an arbitrary function of integration. Substitution of this result in the other equation of motion then yields the required constraints on the velocity field. Thus we have

$$-\frac{v'}{h}\left(-\frac{xv'}{h}+u_0\right)+v\left(-\frac{xv''}{h^2}+\frac{u_0}{h}\right) = -\frac{dP}{dx}+\nu\left(-\frac{xv'''}{h^3}+\frac{u_0''}{h^2}\right),$$

so that

$$P(x) = \tfrac{1}{2}k\frac{x^2}{h^2}+l\frac{x}{h}+m,$$

where k, l, m, are constants. By a suitable choice of origin of x we may take $l = 0$ (provided $k \neq 0$), and the equations for the velocity field are then

$$\left.\begin{aligned} vv''-v'^2-\frac{\nu}{h}v''' &= k,\\ vu_0'-v'u_0-\frac{\nu}{h}u_0'' &= 0.\end{aligned}\right\} \tag{1}$$

The boundary conditions for these equations are

$$v(-1) = V_1, \quad v(1) = V_2, \tag{2}$$

$$v'(-1) = 0, \quad v'(1) = 0, \tag{3}$$

and, if the walls are at rest,

$$u_0(-1) = 0, \quad u_0(1) = 0.$$

Whether or not it is proper to regard this last pair as no-slip conditions, or even their validity, depends to some extent on the nature of the porous material. However, for the purpose in hand, which is really an exercise in boundary-layer theory, the no-slip interpretation seems justified and we take the solution for u_0 to be $u_0 \equiv 0$. (The possibility of eigensolutions has not been examined, largely because the principal problem, that of determining v, is independent of u_0.) The same ambiguity does not arise in connexion with the boundary conditions (3). Here we are merely asserting that there is some condition on the tangential component of velocity at the wall and that it is independent of x. For brevity we again refer to it as a no-slip condition.

The four boundary conditions (2) and (3) determine the constant k and the three constants of integration in the solutions of (1). The problem is thus to find the asymptotic form of these solutions as $\nu \to 0$ for fixed values of V_1 and V_2.

3. Principles for determining the asymptotic solution

3.1. *Possible positions of the viscous layers*

In steady flow, internal viscous layers must obviously coincide with stream surfaces of the asymptotic flow for large Reynolds numbers. The essence of the matter is that there is no mechanism to hold a layer steadily in position against the convective effect of a normal velocity component (unless, of course, the layer is supported by a porous boundary, in which case it is not internal). Also, in view of the geometrical similarity of the flow under consideration at different values of x, the viscous layers must coincide with the co-ordinate surfaces $y =$ constant. The condition that these two kinds of surfaces should coincide is that $v = 0$, and it is in this sense that the positions of the internal layers depend on particular solutions of (1), that is, on the boundary conditions.

We thus start with the idea that, as $\nu \to 0$ for fixed values of V_1 and V_2, the asymptotic solution of (1) is such that the flow is divided into a number of cells which are bounded by either the walls or the planes $v = 0$. In the interior of these cells the solution satisfies the inviscid form of (1). Near the bounding planes the solution may, but does not necessarily, have a viscous structure. At this stage of the argument the flow may consist of any number of cells, and the next step in reducing the number of possibilities is to find the general inviscid solution for v in each of them.

3.2. *The general inviscid solution*

A first integral of the inviscid form of (1), namely

$$vv'' - v'^2 = k, \tag{4}$$

is

$$v'^2 = Av^2 - k, \tag{5}$$

where A is a constant. Then, depending on the signs of A and k, the following possibilities arise:

(i) $k < 0$:

$$A < 0, \quad v = (k/A)^{\frac{1}{2}} \sin\{(-A)^{\frac{1}{2}}\eta + \epsilon\}; \tag{6}$$

$$A = 0, \quad v = (-k)^{\frac{1}{2}}\eta + \epsilon; \tag{7}$$

$$A > 0, \quad v = (-k/A)^{\frac{1}{2}} \sinh\{A^{\frac{1}{2}}\eta + \epsilon\}. \tag{8}$$

(ii) $k = 0,\ A \geqslant 0$:

$$v = \exp\{A^{\frac{1}{2}}\eta + \epsilon\}. \tag{9}$$

(iii) $k > 0,\ A > 0$:

$$v = (k/A)^{\frac{1}{2}} \cosh\{A^{\frac{1}{2}}\eta + \epsilon\}. \tag{10}$$

In each case, ϵ is the second arbitrary constant of integration.

It should be noticed that (9) and (10) have no zeros, so that these solutions can be relevant only to cases in which there are no internal viscous layers. Similarly, (7) and (8) have only one zero, so that these solutions can refer only to cells which are bounded on at least one side by a porous wall. However, (6) has more than one zero, and it is the existence of this solution that continues to make possible, at least at this stage of the argument, the idea of a flow divided into any number of cells.

3.3. *Continuity conditions across viscous layers*

The remaining piece of information that may be deduced from the inviscid equations is that v' must have the same value on both sides of an internal viscous layer. This follows immediately from (5). For k is an exact constant in the original viscous problem and therefore takes the same value in each cell of the asymptotic solution. Then, since $v = 0$ at an internal viscous layer, v' must be continuous through such a layer.

Continuity of v' implies continuity of u, which shows that the internal viscous layers are not shear layers, as they usually are in more general laminar flows at large Reynolds numbers. Apparently they have a weaker structure in which there is a rapid transition in a normal derivative of u rather than u itself (which derivative will emerge shortly).

Finally, at a boundary layer, we take only v to be continuous.

3.4. *Necessary positions of the viscous layers*

The first point to be settled by a consideration of viscous mechanics is whether every zero of v in the list of inviscid solutions need give rise to a viscous layer. In the case of the linear solution (7), the answer is clearly that it need not, since this solution is an exact solution of the full differential equation.

For the remaining two functions (6) and (8), the easiest procedure is to differentiate the differential equation (1) twice, which yields

$$vv^{\mathrm{iv}} - v''^2 - \frac{\nu}{h} v^{\mathrm{v}} = 0. \tag{11}$$

Thus we see that v^{v} must vanish at any point where v and v'' vanish simultaneously. In both of the inviscid solutions (6) and (8), v and v'' vanish simultaneously but v^{v} does not vanish at the same point. Hence, there must be viscous layers at the zeros of (6) and (8), in which there is a rapid transition in at least the fifth derivative (and possibly a lower one, the ambiguity arising from the loss of information through differentiation of the differential equation).

We now have the result that v can change sign in the interior of an inviscid cell only if the solution is (7) and that, in this case, the whole flow consists of a single cell.

3.5. *Structure of the boundary layers*

Near a porous boundary through which fluid flows with velocity V, the boundary-layer approximation to (1) is

$$Vv'' - \frac{\nu}{h} v''' = 0, \tag{12}$$

and the solution for v' (the lowest derivative in which there is a rapid variation across the boundary layer) is

$$v' = K(1 - e^{Vh\eta/\nu}), \tag{13}$$

where K is a constant and the boundary is taken (for the moment) to be at $\eta = 0$.

For present purposes, the most important property of (13) is that a boundary layer is possible only if

$$V\eta < 0. \tag{14}$$

This is the well-known and physically obvious result that a boundary layer can exist on a porous boundary if there is extraction of fluid but not if there is injection of fluid. In the latter case, the inviscid solution in the cell nearest to the boundary must be uniformly valid right up to the boundary and must there satisfy the no-slip condition $v' = 0$.

In the extraction case, the no-slip condition on the inviscid solution may be relaxed, in the usual way, and the constant K must then be taken as the value of v' at the boundary, according to the inviscid solution.

It should be added here that, if $V = 0$, the boundary layer takes the known form for two-dimensional stagnation flow and that, again, a steady boundary layer is possible only if the normal velocity near the boundary is directed towards the boundary.

3.6. *Structure of the internal viscous layers*

According to § 3.3, v' must have the same value $(-k)^{\frac{1}{2}}$ on each side of an internal viscous layer. Hence we attempt to determine the structure of the layer by assuming that the function

$$v \sim (-k)^{\frac{1}{2}}\eta \tag{15}$$

(the layer is again taken to be near $\eta = 0$) provides a uniformly valid first approximation to v and its first derivative. Within the layer we therefore set

$$v = (-k)^{\frac{1}{2}}\eta + \bar{v}, \tag{16}$$

and find from (1) that $\bar{v}$ satisfies the equation

$$(-k)^{\frac{1}{2}}\eta\bar{v}'' - 2(-k)^{\frac{1}{2}}\bar{v}' - \frac{\nu}{h}\bar{v}''' = 0. \tag{17}$$

It then follows that, for a full balance between the inertial and viscous terms, the thickness of the layer must be proportional to $\nu^{\frac{1}{2}}$ and that the amplitude of $\bar{v}$ is left arbitrary. This amplitude is of course fixed by the inviscid solutions on either side of the layer. In fact, according to the inviscid solutions (6), (7) and (8), the first approximation to $\bar{v}$ for small values of η is proportional to η^3, with a constant of proportionality independent of ν (and possibly zero). Hence in the outer parts of the layer, where the viscous and inviscid solutions have to be matched, the amplitude of $\bar{v}$ is proportional to $\nu^{\frac{3}{2}}$; and, owing to the linearity of equation (17), this estimate remains valid throughout the whole layer. Thus the layer is such that $\bar{v}'''$ (and hence v''') remains bounded as $\nu \to 0$. The easiest way to determine the essential role of the layer is then to differentiate (17) twice, to the form

$$(-k)^{\frac{1}{2}}\eta\bar{v}^{\text{iv}} - \frac{\nu}{h}\bar{v}^{\text{v}} = 0, \tag{18}$$

and integrate twice to obtain

$$\bar{v}''' = \alpha\int_0^{\xi}\exp\{\tfrac{1}{2}h(-k)^{\frac{1}{2}}\xi^2\}\,d\xi + \beta \quad (\xi = \eta/\nu^{\frac{1}{2}}), \tag{19}$$

where α and β are constants of integration independent of ν.

The most important property of (19) is, once again, the qualitative constraint arising from the irreversibility of the mechanics. Thus a viscous layer is possible only if $(-k)^{\frac{1}{2}}$ is negative. By (15), this implies that

$$v' < 0 \tag{20}$$

and the condition is again that the normal velocity near the layer is directed towards the layer. When the condition is satisfied, the role of the layer may be regarded as smoothing out a discontinuity in v''' at the boundary between two cells of the asymptotic solution, and the constants α and β must be chosen accordingly.

If it should happen that v''' has the same value on both sides of the layer (and this is so for one particular set of boundary conditions), then $\alpha = 0$ and the essential viscous structure of (19) is lost. Moreover, continuity of v''' across the layer implies continuity of the constant A in (5), and so the inviscid solution in one of the contiguous cells must be the analytic continuation of the solution in the other. Thus, were it not for the results of § 3.4, we might expect a viscous layer to be unnecessary. However, we have seen in § 3.4 that this is possible only for the linear solution (7). For the remaining solutions (6) and (8), a repetition of the preceding analysis with the assumption that

$$v \sim (-k)^{\frac{1}{2}}\eta + \tfrac{1}{6}A(-k)^{\frac{1}{2}}\eta^3$$

is uniformly valid yields the result that v^{v} remains bounded throughout the layer as $\nu \to 0$ with a variation given by

$$\begin{aligned} v^{\mathrm{v}} &= A^2kh\xi^2 - A^2(-k)^{\frac{3}{2}}h^2\xi\exp\{\tfrac{1}{2}(-k)^{\frac{1}{2}}h\xi^2\}\int_0^{\xi}\xi^2\exp\{-\tfrac{1}{2}(-k)^{\frac{1}{2}}h\xi^2\}\,d\xi \\ &\to A^2(-k)^{\frac{1}{2}} \quad \text{as} \quad \xi \to \pm\infty, \end{aligned} \tag{21}$$

provided condition (20) is still satisfied. There are no arbitrary constants in this solution, and the viscous structure disappears only if $A = 0$, which corresponds to the linear solution (7). In this last case it is a simple matter to show that a viscous layer is not only unnecessary but impossible.

4. The asymptotic solution for arbitrary V_1 and V_2

We are now in a position to determine uniquely the asymptotic solution as $\nu \to 0$ for all fixed values of V_1 and V_2. For the most part, the arguments that select the number of inviscid cells and the solutions in each of them depend on the particular combination of signs of V_1 and V_2 under consideration. But there is one important result which is common to all possibilities:

$$v \text{ can have at most one zero.} \tag{22}$$

This follows immediately from the principles of § 3. For, if v has more than one zero, the inviscid solution in the cell bounded by two of them must be (6), this being the only solution with more than one zero. From § 3.4 we then have that both zeros must coincide with viscous layers, and finally, from § 3.6 that one of these layers, the one at which $v' > 0$, is impossible. Hence the asymptotic solution consists of at most two cells, and there is at most one internal viscous layer.

The separate cases are considered below.

4.1. *The case* $V_1 > 0, \quad V_2 \geqslant 0$

By (22), $v > 0$ for all η, and the asymptotic solution consists of a single cell. Moreover, on the left-hand boundary (left and right refer to $\eta < 0$ and $\eta > 0$, respectively; see figure 1), there is injection of a fluid and a boundary layer is impossible, by § 3.5. The inviscid solution must therefore satisfy the boundary conditions

$$v(-1) = V_1, \quad v'(-1) = 0, \quad v(1) = V_2. \tag{23}$$

Now the only inviscid solutions with zero derivatives are (6) and (10), and then, depending on the relative magnitudes of V_1 and V_2, the unique solution satisfying (23) is easily found to be

$$V_2 < V_1, \quad v = V_1 \cos\{\tfrac{1}{2}(1+\eta)\cos^{-1}(V_2/V_1)\}; \tag{24}$$

$$V_2 = V_1, \quad v = V_1; \tag{25}$$

$$V_2 > V_1, \quad v = V_1 \cosh\{\tfrac{1}{2}(1+\eta)\cosh^{-1}(V_2/V_1)\}. \tag{26}$$

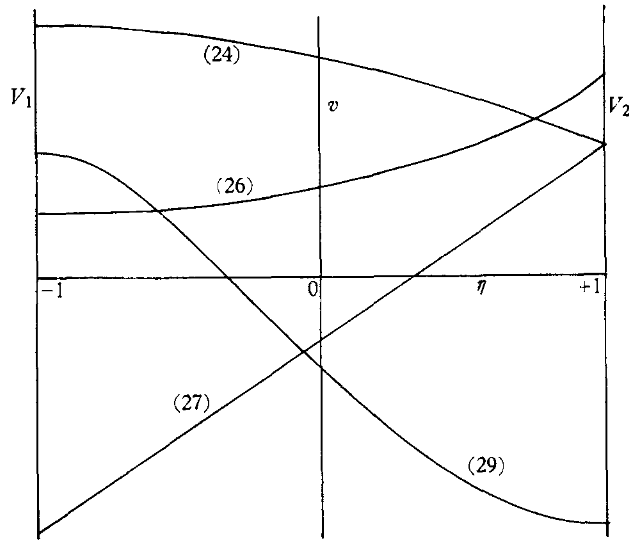

FIGURE 1. The asymptotic solution for v as $\nu \to 0$. The number beside each curve refers to the corresponding equation in the text. The velocity scale is arbitrary.

In (24) the smallest positive values of $\cos^{-1}(V_2/V_1)$ must be taken. In every case there is a conventional 'suction' boundary layer on the right-hand boundary of the type (13).

4.2. *The case* $V_1 < 0, \quad V_2 > 0$

In this case there must be a zero of v at which $v' > 0$. The only possible solution is therefore (7):

$$v = V_1 + \frac{V_2 - V_1}{2h}(1+\eta). \tag{27}$$

On both walls there are suction boundary layers of the type (13).

The solution (27) represents an irrotational flow and is, of course, unique in this respect. That such a flow should develop when fluid is extracted from both boundaries is to be expected; and this provides a more direct method of deriving (27). In fact, Sellars (1955) used essentially the same argument to derive the special case of (27) in which $V_1 = -V_2$.

4.3. *The case* $V_1 = 0, \quad V_2 > 0$

This case represents the transition between those considered in §§4.1 and 4.2. It is especially interesting, not only for the physical results themselves but also for the way in which the solution appears as a one-sided limit as $V_1 \to 0$ in the results of §§4.1 and 4.2.

Direct consideration of the case $V_1 = 0$ leads to the conclusion that there can be no boundary layer on the left-hand wall. For, if there were, we should have $v'(-1) < 0$ (see the remark at the end of § 3.5), and this would imply the existence of a second zero of v, which is known to be impossible. Thus the inviscid solution in the cell adjacent to the left-hand wall must be uniformly valid up to the wall and must there satisfy the two conditions

$$v(-1) = v'(-1) = 0.$$

The only such solution is

$$v \equiv 0, \quad \eta \neq 1. \tag{28}$$

This solution does not satisfy the right-hand boundary condition, and it seems that there must be a boundary layer on this wall, of a kind much more severe than any considered in § 3: a boundary layer which is responsible for all of the mass flow through the porous wall.

The result (28) may also be obtained as the limit of (26) as $V_1 \to 0$. However, it may not be obtained as the corresponding limit of (27), which refers to negative values of V_1. The reason appears to be the nature of the discontinuity in the left-hand boundary condition on the inviscid solution as V_1 passes through zero: a boundary layer is possible if $V_1 < 0$ but not if $V_1 \geqslant 0$. Thus $V_1 = 0$ should be regarded, so to speak, as the case of no injection, rather than no extraction, and the solution is obtained as the corresponding one-sided limit.

It should be emphasized that (28) means that $v/V_2 \to 0$ for $\eta \neq 1$ as $\nu \to 0$, and gives no indication of the actual velocity distribution in the channel. This failure of the asymptotic analysis to produce more than a qualitative result is due, of course, to the fact that viscosity is everywhere important in this case. If V_2 and h are regarded as the fixed standards of velocity and length from which the large Reynolds number is defined, viscosity is important near the right-hand wall because the length scale is short, and important everywhere else because the velocity is small.

4.4. *The case* $V_1 > 0$, $V_2 < 0$

When fluid is injected through both boundaries, neither boundary can support a boundary layer and all of the original boundary conditions must be satisfied by the asymptotic inviscid solution. Thus, on grounds of over-determinacy alone (the inviscid equations are of lower order) one would expect an internal viscous layer in this case. That this is so follows immediately from the principles of § 3. The boundary conditions require that there should be a zero of v in the interior of the channel, and the only inviscid solution which has this property and also satisfies the no-slip condition at a wall is (6). The zero therefore coincides with a viscous layer, by § 3.4, and there are two inviscid cells in the asymptotic solution.

The position of the layer follows from the condition that v' must have the same value on both sides. Thus we find:

$$\left.\begin{aligned} v &= V_1 \sin\frac{\pi(\eta_0-\eta)}{2(\eta_0+1)} \quad (\eta < \eta_0), \\ v &= V_2 \sin\frac{\pi(\eta-\eta_0)}{2(1-\eta_0)} \quad (\eta > \eta_0), \end{aligned}\right\} \quad \eta_0 = \frac{V_1+V_2}{V_1-V_2}. \tag{29}$$

The viscous layer at $\eta = \eta_0$ is of the type (19).

When $V_1 = -V_2$, $\eta_0 = 0$ and the two halves of (29) reduce to the same form. There is then, apparently, a single inviscid solution satisfying all the boundary conditions of the problem, a result which enabled Taylor (1956) and Yuan (1956) to obtain the solution for this special case without considering the possibility of viscous layers. However, this is a very special case. There is still a viscous layer, now at the centre of the channel and of the weaker form (21), and it is a consequence of symmetry that the solution in one inviscid cell is the analytic continuation of the solution in the other, as mentioned in § 3.6.

During part of the period of preparation of this paper, the author was a visitor at the Guggenheim Aeronautical Laboratory of the California Institute of Technology, and he is especially grateful to Professor Paco Lagerstrom and Dr Saul Kaplun for many stimulating discussions about the nature of laminar flow at large Reynolds numbers.

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